Math 2D Multivariable MT 2 Sample Problems

Aaron Chen

****This is in no way indicative of the actual exam!!!****

These problems are just to test yourself with some problems that didn't come out of the book. I'm just using miscellaneous exams from other universities. A few of them are strange or difficult. They're intended mainly as thinking exercises - to think in different perspectives.

***Your best study materials are homeworks, quizzes, and Hamid's Practice Problems!**

1. (a) What is $\frac{d}{dt}F(t, 3t^2, 5)$? Here, F is a differentiable function of 3 variables. Express your answer in terms of the partial derivatives of F.

Solution. We see here that

$$\frac{d}{dt}F(t, 3t^2, 5) = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt}.$$

Here, we see that

$$\frac{dx}{dt} = \frac{d}{dt}[t] = 1, \quad \frac{dy}{dt} = \frac{d}{dt}[3t^2] = 6t, \quad \frac{dz}{dt} = \frac{d}{dt}[5] = 0.$$

Thus,

$$\frac{d}{dt}F(t, 3t^2, 5) = F_x(t, 3t^2, 5) \cdot 1 + F_y(t, 3t^2, 5) \cdot 6t.$$

(b) Find the normal line to the surface $(x + y)^3 - (y + z^2)^5 = 9$ at the point (4,-2,1).

Solution. We need to find the gradient of this level surface $F(x, y, z) = (x + y)^3 - (y + z^2)^5 = 9$,

$$\nabla F(x, y, z) =< 3(x + y)^2;\ 3(x + y)^2 - 5(y + z^2)^4;\ -2z(y + z^2)^4 >$$

and at $(4, -2, 1)$, after evaluating and simplifying,

$$\nabla F(4, -2, 1) =< 12,\ 7,\ -2) > .$$

This is the direction of our normal line, and it is to go through the point (4,-2,1) so we have

$$\frac{x - 4}{12} = \frac{y + 2}{7} = \frac{z - 1}{-2}$$

as our symmetric equation of the normal line. (You can also write it in parametric).

2. Find the maximum and minimum values of the function $f(x, y) = (x^2 + 2y^2)^{1/2}$ on the domain (it is a disk) $D = \{(x, y) : (x - 1)^2 + y^2 \leq 9\}$.

Solution. First, we need critical points. Here,

$$\nabla f(x, y) =< \frac{2x}{2\sqrt{x^2 + 2y^2}}\ ,\ \frac{4y}{2\sqrt{x^2 + 2y^2}} > .$$

There's actually no place that it is zero - it looks like the origin (0,0) is a critical point, but $\nabla f(0, 0)$ is actually undefined since we divide by zero. Nonetheless, *points where ∇f is undefined are still critical points.* So, we need to tabulate (0,0), and we see that at (0,0),

$$f(0, 0) = 0.$$

(Note: Since f is the sum of squares, we should expect this to be the absolute minimum).

Second, we maximize along the boundary. The boundary here is $(x-1)^2+y^2=9$. In other words, $y^2 = 9-(x-1)^2$. Therefore on the boundary, our single variable functions are

$$g_{\pm}(x) = f(x,\ \pm\sqrt{9-(x-1)^2}) = (\ x^2 + 2(9-(x-1)^2)\ )^{1/2}$$

and simplifying,

$$g_{\pm}(x) = (-x^2+4x+16)^{1/2} \ \text{ from both of } y = \pm\sqrt{9-(x-1)^2}.$$

When we try maximizing this boundary function,

$$g'(x) = \frac{-2x+4}{(16+4x-x^2)^{1/2}}$$

has a zero at $x=2$. (At $x=2$, $y^2 = 9-(1)^2 = 8$). Therefore,

$$g(2) = f(2, \pm\sqrt{8}) = \sqrt{20}.$$

Lastly, even though it doesn't look like we have boundaries to check, **we have to check** $x=-2$ **and** $x=4$ **boundary points**. In other words the edges of the circles where $y=0$. This is because our $g(x)$ is technically defined on $[-1,1]$ from the functions $y=\pm\sqrt{9-(x-1)^2}$. At these points, we see

$$g(-2) = f(-2,0) = \sqrt{4}, \quad g(4) = f(4,0) = \sqrt{16}.$$

Both these values are less than $\sqrt{20}$.

Compiling all our results, we see that

$$\text{Absolute Max: } f(2,\pm\sqrt{8}) = \sqrt{20}, \quad \text{Absolute Min: } f(0,0)=0.$$

3. Let

$$f(x,y) = \frac{x+y}{\sqrt{x^2+y^2}}, \quad (x,y) \neq (0,0).$$

(a) Write the equation in polar coordinates.

Solution. We use $x = r\cos\theta, y = r\sin\theta$, so

$$f(r,\theta) = \frac{r\cos\theta + r\sin\theta}{r} = \cos\theta + \sin\theta.$$

(b) Does the limit as $(x,y)\to(0,0)$ of $f(x,y)$ exist? (Explain why or why not).

Solution. When we equivalently take $r\to 0$, we see that

$$\lim_{r\to 0} f(r,\theta) = \cos\theta + \sin\theta.$$

The limit depends on the angle θ so the limit does NOT exist.

Alternatively, you can try paths like $x = my$ (or y $= mx$) and see we get something like

$$\lim_{y\to 0} f(my,y) = \frac{y(m+1)}{y\sqrt{m^2+1}} = \frac{m+1}{\sqrt{m^2+1}}$$

which depends on m so again it does not exist.

(c) Does the graph of f contain any straight lines? (Perhaps with a single point, namely the origin, missing). Explain why or why not.

Solution. There are actually some straight lines. For example, we see that

$$f(r, \theta) = \cos\theta + \sin\theta.$$

For fixed θ values, we see that f is constant. But, this just means that f is a "horizontal line" at any fixed angle θ from the x-axis.

4. Every equation of the form $ax^2 + by^2 + cz^2 = d$ where a, b, c, d are <u>nonzero</u> real numbers, will describe one of the following: A hyperboloid of one sheet, a hyperboloid of two sheets, an ellipsoid, or an "empty" surface. Explain how to tell, given the numbers a, b, c, d what kind of surface it describes. (Mainly, when do we get the different hyperboloids, cones, and ellipsoids).

Solution. FIRST, let us assume that $d > 0$. We have the following:
If $a, b, c > 0$: We get an ellipsoid.
If only one of a, b, c is negative: We get a hyperboloid of one sheet.
If only one of a, b, c is positive: (In other words, only two of them are positive). We then get a hyperboloid of two sheets.
If all are negative: We get the "empty" surface because the equation doesn't make sense.

SECOND if $d < 0$, we would have the following, it's "backwards" the other case: If $a, b, c > 0$: We get an empty surface this time.
If only one of a, b, c is negative: We get a hyperboloid of two sheets.
If only one of a, b, c is positive: (In other words, only two of them are positive). We then get a hyperboloid of one sheets.
If all are negative: We get the ellipsoid.
The idea is that for these surfaces, we divide everything by -1 and it is like when $d > 0$. (Be sure to do this in actual problems, too!). For example,

$$x^2 - 2y^2 + 6z^2 = -5 \iff -x^2 + 2y^2 - 6z^2 = 5.$$

5. All I'm going to tell you about this differentiable function f on $\mathbb{R}^2$ is that

$$\nabla f(2,3) = < 4, -1 >, \quad f(2,3) = 7.$$

Answer the following questions for which enough information is given, and explain why not enough information is given for the others.
(a) Is $(2, 3)$ a local maximum, a local minimum, or neither, for f?

Solution. It is neither because the gradient is not zero.
(WARNING AGAIN: Even if $\nabla f(2, 3) = 0$, we cannot tell if it's a local max or min - we need the 2nd derivative test.)

(b) Find the best approximation you can for $f(2.04, 2.99)$. (It may help to do this after part (c)).

Solution. We can estimate this by

$$
\begin{aligned}
f(2.04, 2.99) &\approx f(2,3) + f_x(2,3) \cdot (2.04 - 2) + f_y(2,3) \cdot (2.99 - 3) \\
&= 7 + 4(0.04) - 1(-0.01) = 7 + 0.16 - 0.01 \\
&= 7.15
\end{aligned}
$$

(c) There's one point on the graph of f where we have enough information to find the equation of the tangent plane. What are the coordinates of that point? What is the tangent plane's equation?

Solution. We can do this at $(x, y) = (2, 3)$. The coordinates of that point are $(2, 3, 7)$ when we now include the $f(2, 3)$ vaue. The tangent plane at this point is hence defined to be

$$
\begin{aligned}
z - z_0 &= f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\
z - 7 &= 4(x - 2) - 1(y - 3).
\end{aligned}
$$

In otherwords,

$$f(x, y) = z \approx 7 + 4(x - 2) - (y - 3).$$

(d) Is $f(2, 4) \geq 7$?

Solution. We do not have enough information for this. For example, it could do some crazy things on the way from $y = 3$ to $y = 4$.

(e) Find $g'(1)$ where $g(t) = f(2t^3, t^3)$.

Solution. This is an application of chain rule, where we see that

$$g'(t) = f_x(2t^3, t^3) \cdot \frac{d}{dt}[2t^3] + f_y(2t^3, t^3) \cdot \frac{d}{dt}[t^3].$$

At $t = 1$, we see that $(2t^3, t^3) = (2, 1)$ but we have no information about this point. So we actually can't do this!

For practice: If instead $g(t) = f(2t^3, 3t^2)$, then we could do it because $(2t^3, 3t^2) = (2, 3)$ when $t = 1$. We would then have

$$
\begin{aligned}
g'(1) &= f_x(2,3) \cdot 6(1)^2 + f_y(2,3) \cdot 6(1) \\
&= 4 \cdot (6) - 1 \cdot (6) = 18.
\end{aligned}
$$

6. Show that the following limit does not exist:

$$\lim_{(x,y)\to(1,1)} \frac{x + 2y - 3}{x + y - 2}.$$

Solution. One way is to take line paths, like $y-1=m(x-1) \iff y=m(x-1)+1$. We have to use these lines because the point is not the origin. With this, we see that

$$\lim_{x\to 0}\frac{x+2[m(x-1)+1]-3}{x+m(x-1)+1-2}=\lim_{x\to 0}\frac{x(2m+1)-2m-1}{x(m+1)-1}$$

in the limit we only take the highest powers of x,

$$=\frac{2m+1}{m+1}$$

and we see this limit depends on the slope m so it does not exist.

Alternatively: Another way is to note this is the same as

$$\lim_{(x,y)\to(1,1)}\frac{x+2y-3}{x+y-2}=\lim_{(x,y)\to 0}\frac{(x+1)+2(y+1)-3}{(x+1)+(y+1)-2}=\lim_{(x,y)\to(0,0)}\frac{x+2y}{x+y}$$

and this can be similarly shown to not exist. Even easier,
One path: $x=0, y\to 0$, we get

$$\lim_{y\to 0}\frac{0+2y}{0+y}=2.$$

Another path: $x\to 0, y=0$, we get

$$\lim_{x\to 0}\frac{x+0}{x+0}=1.$$

$2\neq 1$ so the limit can't exist because it doesn't equal on these two paths.

7. Suppose for some function $f(x,y)$ we have that

$$f_x=\frac{\partial f}{\partial x}=\frac{1}{x},\quad f_y=\frac{\partial f}{\partial y}=\frac{1}{y}.$$

Let $x=r\cos\theta,\ y=r\sin\theta$. Calculate $\partial f/\partial r$ and $\partial f/\partial\theta$.

Solution. From chain rule, we have that

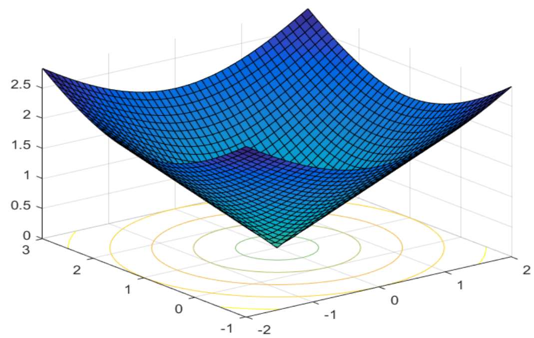
$$\frac{\partial f}{\partial r}=f_x\frac{\partial x}{\partial r}+f_y\frac{\partial y}{\partial r}=\frac{1}{x}\cos\theta+\frac{1}{y}\sin\theta.$$

You could plug in $x=r\cos\theta$ and $y=r\sin\theta$ if you wish.

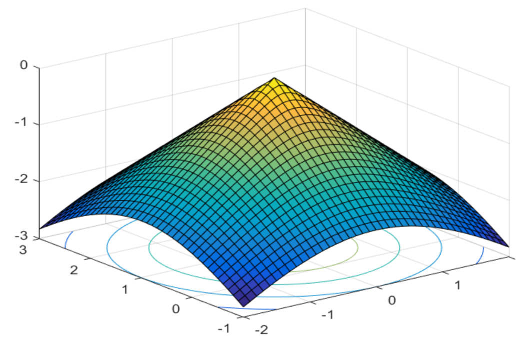
$$\frac{\partial f}{\partial\theta}=f_x\frac{\partial x}{\partial\theta}+f_y\frac{\partial y}{\partial\theta}=\frac{1}{x}\cdot(-r\sin\theta)+\frac{1}{y}r\cos\theta$$

8. (a) Sketch the surface $x^2+(y-1)^2=z^2$.

Solution. This is a cone opening in the z axis, centered at $(0,1,0)$ though.



Top part of the cone.



Bottom part of the cone.

For whatever reason, I couldn't plot it on the same graph. They're supposed to touch tops. The direction that the cone opens is the z-direction.

(b) Find the tangent plane at $(4,4,5)$.

Solution. We first have our level surface as $F(x,y,z) = x^2 + (y-1)^2 - z^2 = 0$. Now we construct the tangent plane, first we need the gradient,

$$\nabla F(x,y,z) =< 2x, 2(y-1), -2z >$$

and at the point (4,4,5), we see that $\nabla F(4,4,5) =< 8, 6, -10 >$. This is the normal to our tangent plane (and to our surface). Therefore, our tangent plane equation is

$$8(x-4) + 6(y-4) - 10(z-5) = 0.$$

9. Find the maximum and minimum values of the function $f(x,y) = (x-1)^2 + (y-1)^2$ on the unit disk $x^2 + y^2 \leq 1$. [Rather difficult.]

Solution. This question is actually a very difficult, but doable absolute max/min problem, that is if you go through the motions of finding the gradient, finding boundary extrema, etc.
I think this problem is meant to be solved with some qualitative understanding.

If we called $z = f(x,y)$, then we'd have $z = (x-1)^2 + (y-1)^2$ which is an elliptic paraboloid. The $z = k$ traces, or level sets, or contours (whichever you prefer) are all circles centered at (1,1). Or, it may also be helpful to note that f is representing the distance squared from (x,y) to (1,1).

Since $z = f(x,y)$ is the sum of squares, it is minimized at the closest point on the unit disk to $(1,1)$ which is at $(x,y) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, so

$$\text{Absolute Min: } f(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = 2\left(\frac{1}{\sqrt{2}} - 1\right)^2 = 2\left(\frac{1-\sqrt{2}}{\sqrt{2}}\right)^2 = 5 - 2\sqrt{2}.$$

When is it maximized though? We see from the traces, it should be the point in the unit disk that is the farthest from (1,1). That would be the point $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$. The value would then be

$$\text{Absolute Max: } f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = 2 \cdot \left(-1 - \frac{1}{\sqrt{2}}\right)^2 = 5 + 2\sqrt{2}.$$

10. Let $f(t) =< 3t^2, 2t^3 >$. Find the length of $f(t)$ between $0 \le t \le 2$.

Solution. First, we need

$$f'(t) =< 6t, 6t^2 > .$$

Then the length is given by

$$L = \int_0^2 \sqrt{(6t)^2 + (6t^2)^2} dt = \int_0^2 \sqrt{36(t^4 + t^2)} dt$$

$$= \int_0^2 6t\sqrt{t^2 + 1} dt$$

Let $u = t^2 + 1, du = 2tdt \iff dt = \frac{1}{2t} du$, and then $u = 1$ when $t = 0$ and $u = 5$ when $t = 5$,

$$L = \int_1^5 3\sqrt{u}\ du = 3 \cdot \frac{2}{3} u^{3/2}\Big|_1^5 = 2\ (5\sqrt{5} - 1).$$

11. Consider the function $f(x, y) = x^4 + y^4 + x^2y^2 - xy + 3$. Find the direction of steepest descent from the point $(1, 2)$. (Not part of the question, but: What is the rate of descent?)

Solution. We need to first get the gradient,

$$\nabla f(x, y) =< 4x^3 + 2xy^2 - y,\ 4y^3 + 2x^2y - x >$$

where at the point (1,2), we have

$$\nabla f(1, 2) =< 4 + 8 - 2,\ 32 + 4 - 1 > = < 10, 35 > .$$

The direction of steepest *descent*, as opposed to ascent, is hence the negative direction,

$$\mathbf{u} = \frac{1}{\sqrt{100 + 35^2}} < -10, -35 > = \frac{1}{\sqrt{1275}} < -10, -35 > .$$

The rate of descent is $|\nabla f(1, 2)| = \sqrt{1275}$

12. Suppose we are given $z^2 + 4x + 4y^2 - 24y = x^2 + 2z$. Convert this to the standard form of a quadratic surface. What kind of surface is it? Sketch it.

Solution. To convert to standard form, we need to complete squares. Let us first get everything to the left side,

$$z^2 + 4y^2 - x^2 + 2z - 24y + 4x = 0.$$

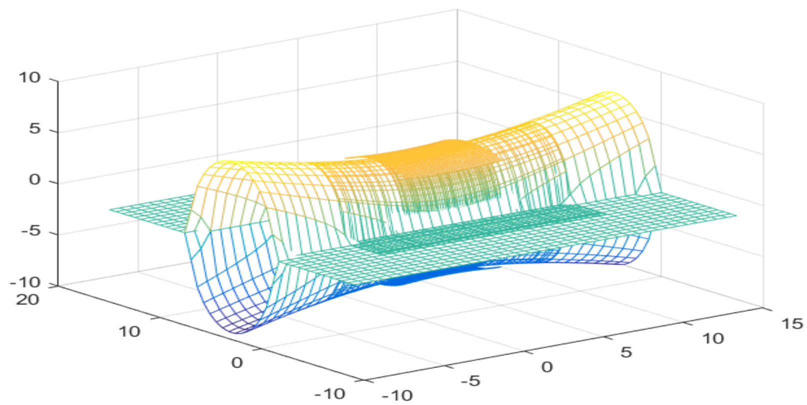
Completing the squares now, and keeping track of the constants we added on both sides,

$$(z + 1)^2 + 4(y - 3)^2 - (x - 2)^2 = 1 + 36 - 4.$$

In other words, we have

$$-(x-2)^2 + 4(y-3)^2 + (z+1)^2 = 33.$$

This is a hyperboloid of 1 sheet, I tried to graph it in Matlab,



Hyperboloid.

The direction that it opens is the x direction. Ignore the excess flat things...

13. Calculate the arclength of the curve

$$r(t) =< 2t^3/3 + 1/3, t+7, t^2+1 >, \quad -1 \le t \le 2.$$

Solution. Again, we need the derivative,

$$r'(t) =< 2t^2,\ 1,\ 2t > .$$

Then the arclength formula states

$$L = \int_{-1}^{2} \sqrt{(2t^2)^2 + 1^2 + (2t)^2} dt = \int_{-1}^{2} \sqrt{4t^4 + 4t^2 + 1} dt$$

$$= \int_{-1}^{2} \sqrt{(2t^2+1)^2} dt = \int_{-1}^{2} (2t^2+1) dt$$

$$= \left[\frac{2t^3}{3} + t \right] \Bigg|_{-1}^{2} = \frac{16}{3} + 2 - \frac{-2}{3} - (-1)$$

so thus

$$L = 3 + \frac{18}{3} = 9.$$

14. What I had intended for Morning Quiz 4: Determine if the following limit exists,

$$\lim_{(x,y)\to(0,0)} \frac{x^2 \sin^2(y)}{x^2 + 2y^2}.$$

Solution. We see that since $\sin(y) \approx y$ when $y \approx 0$, we expect this limit to exist because the power of r in the numerator is 2+2=4 and the power in the denominator is just 2. Using polar,

$$\lim_{r\to 0} \frac{r^2 \cos^2\theta \sin^2(r\sin\theta)}{r^2(\cos^2\theta + 2\sin^2\theta)} = \lim_{r\to 0} \frac{\sin^2(r\sin\theta)\cos^2\theta}{(1+\sin^2\theta)}.$$

We expect this to go to zero independently of θ since $\sin(0) = 0$. To do this rigorously, we utilize the Squeeze Theorem. First, we see that $\cos^2\theta \le 1$ in the numerator and $1 + \sin^2\theta \ge 1$ in the denominator. Also, $\sin^2(r\sin\theta) \le r^2\sin^2\theta$ since $|\sin(t)| \le |t|$. And $r^2\sin^2\theta \le r^2$ since $\sin^2\theta \le 1$. These give us an *upper bound on the limit.* Lastly, everything is squared, so the limit must be ≥ 0 as a *lower bound on the limit.* Thus, for any value of θ, when $r \approx 0$,

$$0 \le \frac{\sin^2(r\sin\theta)\cos^2\theta}{1+\sin^2\theta} \le \frac{r^2\sin^2\theta \cdot 1}{1} \le r^2$$

so in the limit as $r \to 0$, we see that

$$\lim_{r\to 0} 0 = 0 \le \lim_{r\to 0} \frac{\sin^2(r\sin\theta)\cos^2\theta}{1+\sin^2\theta} \le \lim_{r\to 0}\ r^2 = 0.$$

So by the Squeeze Theorem, this limit exists (independently of θ) and equals zero.

15. Combining my failed Morning Quiz 4 and Afternoon Quiz 4:
Determine the set of points that f is continuous, where

$$f(x,y) = \begin{cases} \frac{x^2\tan^2(y)}{x^3+2y^3} & (x,y) \ne (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$$

Solution. This function is dividing by zero if $x^3 + 2y^3 = 0 \iff x^3 = 2y^3 \iff x = -y\sqrt[3]{2}$. Therefore, it is not continuous at these points.

This function also goes to infinity when $\tan(y)$ goes to infinity, in other words it is discontinuous when $y = \pm\pi/2, \pm 3\pi/2, ...$ Therefore, the set S that it actually is continuous is

$$S = \{(x,y) \in \mathbb{R}^2 \text{ such that } x \ne -y\sqrt[3]{2} \text{ and } y \ne \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2, etc.\}.$$

16. See problems from Sections 13.1,2,4 and 14.1 as those were not touched upon too much.